The Case Against Eliminating Large Denomination Bills*

Joshua Hendrickson† Jaevin Park‡

July 29, 2020

Abstract

When large denomination bills are preferred in illegal activities, what is the optimal policy response? We construct a dual currency model where illegal activity can be reduced by modifying the payment environment. In our model, legal (goods) traders are indifferent between small and large bills, but illegal (goods) traders face a lower transaction cost of using large bills in comparison to small bills because it is easier to conceal. We show that eliminating large bills can reduce illegal trade and its associated social cost. However, this pooling equilibrium is sub-optimal because the government can collect more seigniorage by allowing illegal traders to use large bills with a lower rate of return. When the transaction cost of using small bills for illegal traders is sufficiently large, a separating equilibrium, where legal traders use small bills and illegal traders use large bills, can maximize welfare by making an implicit transfer from the illegal traders to the legal traders.

Key Words: illegal activities, dual currency, seigniorage, separating equilibrium

JEL Codes: D62, E26, E52

---

*We are grateful to David Andolfatto, Jonathan Chiu, Julio Garin, Todd Keister and Yu Zhu for their useful comments and suggestions. We would also like to thank the seminar participants at the University of Mississippi and at the Midwest Macro conference held at the University of Georgia.

†Department of Economics, University of Mississippi. E-mail: jhendrl@olemiss.edu

‡Department of Economics, University of Mississippi. E-mail: jpark21@olemiss.edu
“Yet again, it points to the fact that demand for large-denomination notes is qualitatively different from that for small-denomination notes. ... The 100-dollar bill and the 500-euro note, for example, are relatively unimportant in everyday retail transactions.” – Kenneth Rogoff, *The Curse of Cash*

1 Introduction

Despite the existence of alternative means of payment, including those that earn a higher rate of return, cash is still used in many transactions. One reason individuals might choose to use cash rather than other assets is that cash is free of record-keeping. For example, two parties to a transaction might prefer to exchange cash for goods and services to avoid the costs associated with third-party processing. Alternatively, individuals who are engaged in illegal activity might also prefer cash because of the lack of record-keeping involved.

If cash is used for illegal activity, and we want to discourage this activity, then a natural question is whether such activity can be reduced or eliminated by modifying the payment environment. Rogoff (2016) claims that large denomination bills are widely used in the underground economy such as the drug trade, bribes, crime, and money laundering. He argues that policymakers should eliminate large denomination bills in order to reduce illegal transactions. Some countries have recently ceased issuing large denomination bills.\(^1\) Williamson (2017) agrees that the gain from reforming the currency system could be significant, but argues that the elimination of a specific bill could result in a reduction of seigniorage to the central bank.\(^2\) The central bank can earn seigniorage by generating a strictly positive inflation rate. Moreover, raising the inflation rate reduces cash transactions as the rate of return on cash decreases. So eliminating cash or large denomination bills, instead of raising the inflation rate, may not be the optimal solution when we consider the benefit of seigniorage to the government or central bank. There-

---

\(^1\)For example, Singapore demonetized its largest denominated bills in 2014 and the European Central Bank (ECB) withdrew the 500-euro bills in 2016.

\(^2\)Williamson (2017) also argues that eliminating large denomination bills can be costly to those who use these large denomination bills for legal purposes. If large denomination bills have been actively used in legal transactions, there is at least a proportion of people who choose to use these bills instead of other payment methods. So given elimination of large denomination bills, they would have to use an alternative payment method that is more costly.
fore, even if this sort of demonetization policy is effective in reducing illegal activities, it is not obvious that this is the welfare-maximizing policy.

In this paper we consider the role of seigniorage to determine whether eliminating large denomination bills can be the welfare-maximizing policy. For example, if the behavior of illegal traders creates a social cost, then a Pigouvian tax on illegal traders or a transfer from the illegal traders to legal traders might be the most efficient solution. However, since those trading illegal goods have an incentive to keep their identity private, this would seem to imply that either an explicit tax or a transfer is infeasible. Nonetheless, if one could design a policy that would differentiate individuals by their types, then such a tax or a transfer could occur. One way to produce this type of policy would be to have a dual currency system with the exchange rate between the currencies determined by the market. Suppose that one type of currency is preferred by illegal traders. Policymakers could vary the rates of return between the two currencies such that illegal traders choose to hold one of the currencies whereas legal traders choose to hold the alternative. By doing so, policymakers are able to treat traders differently even under private information and can use this relative difference in the rates of return to generate a transfer via seigniorage from illegal traders to legal traders.

Since this type of policy engineers the sort of transfer by which the social cost can be paid by the creator, it might be preferable to eliminating the currency used by illegal traders. In other words, by eliminating large denomination bills, policymakers can reduce illegal activity, but this might come at the cost of losing the policy outlined in our example. It is important to consider whether eliminating large denomination bills is a desirable policy and whether designing policy to engineer a transfer from those engaged in illegal activity to those engaged in legal activity is the most preferable solution.

The purpose of this paper is to use a model, in which money is essential, to examine payment choices when some trades are illegal. To do so, we use a modified version of the monetary search model of Rocheteau and Wright (2005), in which *ex ante* different individuals trade in the decentralized places. We assume that some fraction of agents engage in illegal
trade, which generates a social cost.

The denomination structure of a currency system is constructed to trade efficiently by using various denomination bills together.\(^3\) However, since specific denominations are preferred in some types of transactions, we can also consider the various denomination bills as different types of payment methods.\(^4\) In this respect our model addresses the issue related to denomination size by considering issues of hiding illegal transactions. In the model, we assume that small denomination bills are more costly than large denomination bills for illegal traders to hide their transactions.\(^5\) Furthermore, we allow the exchange rate between small and large bills to be determined by the market, rather than be arbitrarily fixed by the government.

Our model is therefore capable of answering an important policy question. Suppose that there are conditions in which people engaged in illegal trade prefer to use large denomination bills in equilibrium. What is the optimal policy response? Should we eliminate large denomination bills? Or is there a better solution?

The model produces a couple of important results. First, the model shows that eliminating large bills cannot maximize welfare as long as there is an additional transaction cost for illegal traders in using small bills. In the model, phasing out large bills can reduce the amount of illegal trade, but seigniorage revenue is collected inefficiently with the transaction cost of small bills. Allowing illegal traders to use large bills with the lower rate of return can both reduce illegal trade and collect the inflation tax efficiently.

Second, if the transaction cost of large bills for illegal traders is sufficiently small, welfare is maximized in a separating equilibrium in which illegal traders use large bills and legal traders use small bills. Since illegal trades create a social cost, the government can generate seigniorage from the illegal traders by setting low rate of return on large bills and providing implicit transfers to legal traders by setting high rate of return on small bills.\(^6\) Then the amount of illegal trade is

---

\(^3\)Lee et al. (2005) shows that the denomination structure of currency is based on the carrying costs of cash.

\(^4\)In reality, small denomination bills are required for some retail market transactions such as bus fare. In addition, certain businesses sometimes request bills less than or equal to a certain value.

\(^5\)In reality, individuals engaged in illegal trade choose large denomination bills not only for the anonymity, but also for its advantage in hiding their illegal transactions.

\(^6\)Gomis-Porqueras et al. (2017) show that perfectly substitutable currency can remain in circulation with dom-
reduced while the amount of legal trade increases. This is a basic application of microeconomics in which activity that has a social cost above its private cost is taxed to produce the socially desirable outcome.

These results provide us with an implication for optimal monetary policy associated with the denomination structure when illegal activity is pervasive. When one type of denomination of bills is preferred in illegal transactions that create a negative externality, the monetary authority can internalize the social cost by reducing the rate of return on that denomination of currency as long as the other alternative bills are costly for these transactions.

Finally, our main result requires implementing a separating equilibrium with different rates of return on small and large bills. The government can implement it by allowing the exchange rate between the small and large denomination bills to fluctuate over time or providing strictly positive interest on small denomination bills. There is some precedent for this. An example is the patacón, which is a small denomination bond issued by the local government in Argentina in 2001, circulates in the economy at the same as pesos and pays an interest annually.7

2 Related Literature

There is a vast literature that studies the underground economy and measures its size. Many papers pay attention to cash as a payment method for the illegal goods transactions.8 Most of the papers address this issue empirically and focus on identifying and/or estimating the demand for cash transactions in the shadow economy.9 Camera (2001) is one of the few papers that explore the role of money for discouraging illicit activity within a search-theoretical model. In his model illegal goods can be produced with a participation externality and he shows that over-provision of money can be associated with a greater pervasiveness of the illegal activities,

---

7 See Torre et al. (2003) for more details on the patacón.
8 See Schneider and Enste (2000) for a literature review on the underground economy in general.
9 Schneider (2017) provides a literature review of the papers that study the relationship between cash and illegal activities such as corruption, crime, and terrorist financing.
which is inefficient given a negative social cost. Our paper also examines how monetary policy can disincentivize illegal activity, but we consider the effect of collecting seigniorage by using a framework with divisible money.

There are some papers that study the transformation from cash to credit or record-keeping for inhibiting illegal activities. Alvarez and Lippi (2017) assess the welfare cost of phasing out cash in the context of a cash inventory model.\textsuperscript{10} Lotz and Vasselin (2018) also study the elimination of cash within a search-theoretical model where e-money competes with cash.\textsuperscript{11} While record-keeping is available in these papers, we focus on a pure currency economy under lack of record-keeping technology.\textsuperscript{12}

Our result is closely related with the results of Kocherlakota and Krueger (1999). In Kocherlakota and Krueger (1999), given a bias for home goods, separate monies signal the agent’s preference and can be essential to achieve an optimal allocation. The idea of separating types with two payment methods is similar to the approach in our paper, but we show that separating monies can be beneficial because of the distributional effect of seigniorage, even with divisible monies and a centralized market.

In the recent literature on seigniorage with dual currencies, Zhang (2014) studies international currency competition and considers the role of seigniorage. Given a counterfeit risk associated with foreign currency and a national-currency-required transaction, currencies are treated asymmetrically by countries. If the currency is used as an international currency, the government considers seigniorage when implementing monetary policy. We also study multiple currency equilibria given heterogeneous preferences, multiple payment methods, and a government budget constraint. However, we focus more on welfare improvement by making transfers in a single economy rather than examining competition between countries.

\textsuperscript{10}In their model, both cash and credit are simultaneously available and credit use depends on the level of cash holdings and a withdrawal cost.

\textsuperscript{11}In their model given a risk of theft, e-money is safer than cash, but requires an installation cost for merchants.

\textsuperscript{12}There are also some papers that study competition between cash and a record-keeping device: Kim and Lee (2010) study cash and debit cards by assuming a proportional dis-utility cost for cash and a fixed record-keeping cost for debit cards. Li (2011) explains that checks are used in big transactions while cash is used in all transactions with a record-keeping cost in checks.
Finally, our paper is related to the dual currency literature. After the seminal work of Kareken and Wallace (1981), dual currency economies have been studied in various types of search-theoretical models in which currency is essential for transactions and agents have a choice over which currencies to accept and use. These search-theoretic frameworks have been primarily used to study the acceptability of currencies in one country with various government policies such as launching (Lotz and Rocheteau (2002); Lotz (2004)), intervention (Aiyagari and Wallace (1997); Li and Wright (1998); Velde et al. (1999)), restrictions (Curtis and Waller (2000), Hendrickson et al. (2016), Hendrickson and Luther (2017)), as well as competition in international currencies (Matsuyama et al. (1993); Trejos and Wright (1996); Zhou (1997); Ravikumar and Wallace (2002); Kiyotaki and Moore (2003)). While these papers concentrate on coexistence of and/or competition among dual currencies, we address the optimal monetary policy in a dual and/or single currency regime.

3 Model

The model is an adaptation of Rocheteau and Wright (2005). Time is discrete and continues forever. Each time period is divided into two subperiods. In the first subperiod, agents interact in a centralized meeting (CM) where a Walrasian market opens. In the second subperiod, agents are matched pairwise to trade in a decentralized meeting (DM).

The model is populated by two different types of individuals, buyers and sellers, and each type is a continuum with unit mass. Buyers want to consume in the DM and produce in the CM. Sellers want to consume in the CM and produce in the DM. There are two possible goods sold in the DM, legal and illegal goods, whereas one good is sold in the CM.¹³ Buyers are subject to a preference shock in every period CM: a fraction of buyers, ρ, want to consume legal goods, whereas the remaining fraction, 1 − ρ, want to consume illegal goods after the realization in the CM.¹⁴

---

¹³ Two types of goods are required to capture the social cost associated with illegal activity explicitly.
¹⁴ Although the fraction ρ is fixed, the intensive margins of legal and illegal good consumption in the DM can be chosen by comparing marginal benefit and cost, respectively. Thus, the aggregate demands for legal and illegal
An individual buyer has an expected lifetime utility of
\[
E_0 \sum_{t=0}^{\infty} \beta^t \left[ -H^j_t + u(x^j_t) \right]
\]
where \( H^j_t \in \mathbb{R} \) is the supply of labor in the \( CM \), \( x^j_t \in \mathbb{R}^+ \) is the quantity of consumption in the \( DM \), \( j \in \{l, i\} \) denotes the individuals type with \( l \) used to indicate legal goods and \( i \) used to stand for illegal goods, \( u(\cdot) \) is the utility generated from consumption in the \( DM \) where \( u', -u'' > 0, u'(0) = \infty, u'(\infty) = 0, \) and \( -\frac{xu''(x)}{u'(x)} = \gamma \leq 1 \). An individual seller has an expected lifetime utility of
\[
E_0 \sum_{t=0}^{\infty} \beta^t [X^j_t - h^j_t]
\]
where \( X^j_t \in \mathbb{R}^+ \) is consumption in the \( CM \) and \( h^j_t \in \mathbb{R} \) is hours worked in the \( DM \).

Buyers and sellers have a linear production technology such that one unit (hour) of work produces one unit of the output goods in the \( CM \) and \( DM \). We assume that sellers are capable of producing both the legal and illegal goods.

Each buyer and each seller is matched with probability one in the \( DM \) and they negotiate the terms of trade: We assume that buyers suggest a take-it-or-leave-it offer to sellers for simplicity. These meetings are anonymous in the sense that buyers and sellers do not know the previous trading histories of their trading partner. So, there is no reason for buyers to conceal their type information in the \( DM \). Therefore, there is no asymmetric information between buyers and sellers in the \( DM \). Sellers produce whichever of the goods is demanded by the buyer.

Along with lack of memory, we also assume limited commitment: no one can be forced to work for others. Therefore, media of exchange are essential for the \( DM \) trades. In the \( DM \), buyers offer the media of exchange to sellers in exchange for goods. After leaving the \( DM \), sellers enter the \( CM \) in the subsequent period where they can consume the \( CM \) goods by selling the media of exchange in the Walrasian market. There are two assets that can be used as media of exchange, coins and paper money. It is important to note that although we will refer to small trades can be reflected in the model.
denomination money as “coins” and large denomination money as “paper money” for ease of exposition, the model need not have a literal interpretation.\footnote{Nonetheless, we do note that there is historical precedent for notes circulating at a premium/discount to coins (Heckscher, 1954, 92).} Moreover, our model and results do not depend on where the line is drawn between large and small denomination bills, only that such a distinction is known and recognizable.\footnote{Rogoff (2016) provides many examples of large-denominations bills that ordinary citizens seldom use, such as the U.S. 100-dollar bill, Japan’s 10,000-yen note, the Eurozone’s 500-euro note, Switzerland’s 1,000-franc note, etc.}

Although the terms of $DM$ trade are not observable, the production of illegal goods can be detected by the public. We assume that illegal buyers can hide their $DM$ trades by paying a cost, which includes a carrying cost to a secret place. For example, illegal traders can rent a tinted minivan to carry their monies secretly.\footnote{This cost differs from the carrying cost for legal transactions in a public area, which may apply for the legal traders.} This hiding cost requires a proportion of their real money holdings in terms of $CM$ goods and the proportion becomes larger when coins are used, i.e. $\delta_m > \delta_n \geq 0$.\footnote{This cost is paid by $CM$ goods, so it only affects the demand for the money holdings without generating additional social loss.} Without paying this cost, illegal transactions are always detected and the corresponding buyer will be in autarky in the remaining periods.

Unlike legal goods, the consumption of illegal goods in the $DM$ creates a social cost, $c(x_t^i)$, where $c’, c” > 0$. This cost reflects the negative effect of illegal activities on the economy, such as health care costs for illegal drug users or a recovery cost for a terrorist attack, which we will assume requires additional government spending.

In order to understand the role of seigniorage in illegal activity, we assume that government spending is supported only by seigniorage revenue. Let $M_t$ and $N_t$ denote the supplies of coins and paper money, respectively, in the period $t$. Also, let $\phi_t$ and $\psi_t$ denote the prices of coins and paper money, respectively, in terms of the $CM$ goods in period $t$. The supplies of coins and paper money are controlled by the government over time. At the beginning of time ($t = 0$), the government makes a strictly positive transfer, $\tau_0$, in the $CM$ such that,

$$\tau_0 = \phi_0 M_0 + \psi_0 N_0 > 0,$$
to introduce the coins and paper money in the economy. In the following periods the government budget constraint for time $t = 1, 2, \ldots$ is given as

$$
\tau_t = S_t - G_t = \phi_t(M_t - M_{t-1}) + \psi_t(N_t - N_{t-1}) - c(x^i_t) \geq 0
$$

where $S_t$ is the real value of the seigniorage revenue in period $t$ and $G_t$ is the government spending to support the social cost by, $G_t = c(x^i_t)$.

### 3.1 Maximization Problem with Kuhn-Tucker conditions

In the model, the seller’s problem is trivial because sellers can only accept or reject the buyer’s offer in the DM. Since each buyer knows their own type in the CM, they choose their asset portfolio to maximize their expected trading gains in the DM. A buyer who consumes legal or illegal goods solves the following problem in the CM of period $t$:

$$
\max_{m^j_t, n^j_t, x^j_t \geq 0} u(x^j_t) - (1 + \delta^m_{m^j_t>0}) m^j_t - (1 + \delta^n_{n^j_t>0}) n^j_t + \tau_t
\quad (1)
$$

subject to the seller’s participation constraint,

$$
\frac{\beta \phi_{t+1}}{\phi_t} m_{t+1}^j + \frac{\beta \psi_{t+1}}{\psi_t} n_{t+1}^j - x_{t+1}^j \geq 0,
\quad (2)
$$

for $j = \{l, i\}$. All quantities in equations (1)-(2) are expressed in units of the CM consumption good in period $t$. In equations (1)-(2), $m^j_t$ and $n^j_t$ denote the real quantities of coin and paper money held by each legal or illegal buyer for $j = \{l, i\}$. Illegal buyers face the proportional costs, $\delta^m > \delta^n > 0$, of using coins and paper money to avoid detection.

From the individual buyers’ problem we have a Lagrangian, $L^j_t$, which is written as

$$
L^j_t = u(x^j_t) - (1 + \delta^m_{m^j_t>0}) m^j_t - (1 + \delta^n_{n^j_t>0}) n^j_t + \lambda_t \left( \frac{\beta \phi_{t+1}}{\phi_t} m_{t+1}^j + \frac{\beta \psi_{t+1}}{\psi_t} n_{t+1}^j - x_{t+1}^j \right),
\quad (3)
$$

We assume that the initial transfer, $\tau_0$, is sufficiently small to make the supply of assets scarce in the model.
for \( j = \{l, i\} \). For each variable, \( x^j_t, m^j_t \) and \( n^j_t \) for \( j = \{l, i\} \), the Kuhn-Tucker conditions are:

\[
(x^j_t) \quad u'(x^j_t) \leq \lambda^j_t; \quad x^j_t \geq 0; \quad x^j_t (u'(x^j_t) - \lambda^j_t) = 0,
\]
\[
(m^j_t) \quad \lambda^j_t \frac{\partial \psi_{t+1}}{\partial x_t} \leq 1 + \delta m^j_t \{m^j_t > 0\}; \quad m^j_t \geq 0; \quad m^j_t (\lambda^j_t \frac{\partial \psi_{t+1}}{\partial m_t} - 1 - \delta m^j_t \{m^j_t > 0\}) = 0,
\]
\[
(n^j_t) \quad \lambda^j_t \frac{\partial \psi_{t+1}}{\partial n_t} \leq 1 + \delta n^j_t \{n^j_t > 0\}; \quad n^j_t \geq 0; \quad n^j_t (\lambda^j_t \frac{\partial \psi_{t+1}}{\partial n_t} - 1 - \delta n^j_t \{n^j_t > 0\}) = 0,
\]

for \( j = \{l, i\} \). Note that if \( \lambda^j_t = 0 \) then \( m^j_t = n^j_t = 0 \) in the third column of equation (7), so that \( x^j_t = 0 \). Since we are interested in the equilibrium allocations which have strictly positive consumption, i.e. \( x^j_t > 0 \), we require \( \lambda^j_t > 0 \) for both \( j = l \) and \( j = i \).

Suppose \( \lambda^j_t > 0 \). We have the first-order conditions for \( x^j_t \) as

\[
u'(x^j_t) = \lambda^j_t, \quad (5)\]

for \( j = \{l, i\} \). Given the rates of return on monies, \( \frac{\phi_{t+1}}{\phi_t} \) and \( \frac{\psi_{t+1}}{\psi_t} \) for all \( t \), we can write the different types of possible combinations of asset holdings for legal and illegal traders, respectively, as

\[
m^j_t > 0, n^j_t = 0 \quad \text{if} \quad \frac{\phi_{t+1}}{\phi_t} > \frac{\psi_{t+1}}{\psi_t},
\]
\[
m^j_t = 0, n^j_t > 0 \quad \text{if} \quad \frac{\phi_{t+1}}{\phi_t} < \frac{\psi_{t+1}}{\psi_t},
\]\n
\[
m^j_t > 0 \quad \text{and} \quad n^j_t > 0 \quad \text{if} \quad \frac{\phi_{t+1}}{\phi_t} = \frac{\psi_{t+1}}{\psi_t}.
\]

Note that the choice of payment method depends on the relative rate of return on paper money and coins. If the rates of return on both paper and coins are equal, then the individual is indifferent between the two. If the rate of return on coins is higher than the rate of return on paper, then the individual will prefer to hold coins, and otherwise he or she will prefer to hold paper money. Note, however, that the effective rate of return that an illegal trader receives on coins is lower than the rate of return for legal traders because legal traders do not have to pay.
a transaction cost to hide the monies.

If buyers choose to use a particular method of payment then the corresponding first column of equation (4) will hold with equality. So the consumption levels of the buyers, $x_j^t$ for $j = \{l, i\}$, are determined by the first column of equation (4) and equation (5), given the rates of return on monies. Then the demands for paper money and coin money, $m_j^t$ and $n_j^t$, can be derived from the binding constraint, equation (2).

In equilibrium, asset markets clear in the CM for all $t$, so that the total demand for coins and paper money are equal to the supply of outstanding coins and paper money, respectively, as

$$\rho m_l^t + (1 - \rho) m_i^t = \bar{m}_t,$$  
$$\rho n_l^t + (1 - \rho) n_i^t = \bar{n}_t,$$  

(8)

where $\bar{m}_t := \phi_t M_t$ and $\bar{n}_t := \psi_t N_t$ are the real supply of paper money and coin money in equilibrium.

Therefore, the government budget constraint is written as

$$\tau_t = \left(1 - \frac{\phi_{t+1}}{\phi_t}\right)\{\rho m_l^t + (1 - \rho) m_i^t\} + \left(1 - \frac{\psi_{t+1}}{\psi_t}\right)\{\rho n_l^t + (1 - \rho) n_i^t\} \geq G_t.$$  

(9)

Note that in stationary equilibrium the real money supplies are constant over time as $\bar{m}_t = \bar{m}_t$; $\bar{n} = \bar{n}_t$ for all $t$. Since the government can control the supply of monies, $M_t$ and $N_t$, over time, in stationary equilibrium the rates of return on monies, $\frac{1}{\mu} := \frac{\phi_{t+1}}{\phi_t}$ and $\frac{1}{\eta} := \frac{\psi_{t+1}}{\psi_t}$, can be considered as monetary policy variables. From now on, we focus on stationary equilibria in which the equilibrium allocations are determined by the monetary policy variables, $\frac{1}{\mu}$ and $\frac{1}{\eta}$.

**Definition 1.** Given a monetary policy $(\mu, \eta)$, a stationary monetary equilibrium consists of the variables $(m^l, n^l, x^l, m^i, n^i, x^i)$ and the multipliers $\lambda^l, \lambda^i$ and the transfer $\tau$, which satisfy the binding constraint (2), the Kuhn-Tucker conditions in equation (4), the first-order conditions in equation (5), market clearing conditions, equation (8), and the government budget constraint, equation (9).

Finally, by adding expected utilities across agents in a stationary equilibrium, we can write
the welfare function as
\[
W = \rho \{u(x^l) - x^l\} + (1 - \rho) \{u(x^i) - x^i\}.
\]

Note that the social cost can be reflected in the welfare function indirectly, because the government budget constraint is restricted further when the social cost increases.\(^{20}\)

### 3.2 Types of Stationary Monetary Equilibria

Given the equilibrium conditions, there are five different equilibrium cases which are feasible for a monetary policy \((\mu, \eta)\). To see this, consider Table 1. For simplicity, let us denote \(\alpha := \frac{1 + \delta_m}{1 + \delta_n} > 1\) as the relative cost for illegal traders to use coins instead of paper money.

<table>
<thead>
<tr>
<th>Legal</th>
<th>Paper only ((\mu &gt; \eta))</th>
<th>Paper and Coins ((\mu = \eta))</th>
<th>Coins only ((\mu &lt; \eta))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paper only ((\alpha \mu &gt; \eta))</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>Illegal Paper and Coins ((\alpha \mu = \eta))</td>
<td>O</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Coins only ((\alpha \mu &lt; \eta))</td>
<td>X</td>
<td>X</td>
<td>O</td>
</tr>
</tbody>
</table>

In the table, an ‘X’ denotes an equilibrium that is not feasible whereas a ‘O’ denotes an equilibrium that is feasible. Which equilibria are feasible depends on the relative rates of return on coins and paper. Note that there is only one separating equilibrium and that is the case in which illegal traders only hold paper money and legal traders only hold coins. In all other equilibria, one of the media of exchange is held by both types.

#### 3.2.1 Pooling equilibrium with paper money \((\alpha \mu > \eta; \mu > \eta)\)

If \(\alpha \mu > \eta\), then both legal and illegal buyers will use paper money, so \(n^l, n^i > 0\) and \(m^l = m^i = 0\) in equilibrium. Since this is a pooling equilibrium with paper money only, there is no exchange rate between paper money and coin money. According to the first-order conditions,

\(^{20}\)In this way we can consider the social cost of illegal trade without comparing it with the marginal utility of illegal trade directly.
\[ u'(x^l) = \frac{\eta}{\beta} \quad \text{and} \quad u'(x^i) = \frac{(1+\delta_m)\mu}{\beta}, \quad x^l > x^i \] holds in equilibrium because illegal traders need to pay an additional transaction cost for hiding trades. By using the first-order conditions and the binding collateral constraints, \( \frac{\beta}{\eta}m^l = x^l \), \( \frac{\beta}{\eta}n^i = x^i \), we can re-write the government budget constraint (9) with the policy variable, \( \eta \) as

\[
\rho \left( \frac{\eta - 1}{\beta} \right) f \left( \frac{\eta}{\beta} \right) + (1 - \rho) \left( \frac{\eta - 1}{\beta} \right) f \left( \frac{(1+\delta_m)\eta}{\beta} \right) \geq g \left( \frac{(1+\delta_m)\eta}{\beta} \right),
\]

where \( f(\cdot) \) is defined as the inverse function of \( u'(\cdot) \) so that \( f' < 0, f'' > 0 \) and \( g(\cdot) := \gamma \circ f(\cdot) \) and \( g' < 0, g'' > 0 \). Note that the seigniorage revenues from the legal and illegal trades are hump-shaped by \( \eta \) in (11), because of \( f' < 0 \). Since it is inefficient to collect the extra inflation tax to support the same amount of government spending, we focus only on the equilibrium allocations where the seigniorage revenue increases in \( \eta \). Since the left-side of (11) increases in \( \eta \) while the right-side of (11) decreases in \( \eta \), there exists \( \bar{\eta} \) which satisfies (11) with equality. Note that the government budget constraint requires \( \eta \geq \bar{\eta} \) in equilibrium. This equilibrium case is shown as the area of (1) in Figure 1.

[Figure 1 here]

3.2.2 Pooling equilibrium with coins (\( \alpha\mu < \eta \))

If \( \alpha\mu < \eta \), then both legal and illegal buyers use coins and \( m^l, m^i > 0 \) and \( n^l = n^i = 0 \) in equilibrium. Since this is a pooling equilibrium with only coin money, there is no exchange rate between paper money and coin money, either. According to the first-order conditions, \( u'(x^l) = \frac{\mu}{\beta} \) and \( u'(x^i) = \frac{(1+\delta_m)\mu}{\beta} \), \( x^l > x^i \) still holds in equilibrium, but the gap between \( x^l \) and \( x^i \) becomes larger because it is more costly for illegal traders to use coins. By using the first-order conditions and the binding collateral constraints, \( \frac{\beta}{\eta}m^l = x^l \), \( \frac{\beta}{\eta}n^i = x^i \), we can also

---

\[ 21 \]Given the CRRA utility function of \( -\frac{u''(x)}{u'(x)^2} = \gamma \leq 1 \), \( u'(\cdot) \) is homogeneous of degree \(-\gamma \) and so \( f(\cdot) \) is homogeneous of degree \(-\frac{1}{\gamma} \).

\[ 22 \]In order to guarantee the feasibility of the equilibrium allocation, we require \( \eta \leq \hat{\eta} \) where the seigniorage revenue from the legal trade is maximized by \((\hat{\eta} - 1)f'(\frac{\hat{\eta}}{\beta}) + f(\frac{\hat{\eta}}{\beta}) = 0 \).
re-write the government budget constraint (9) with the policy variable, \( \mu \) as

\[
\rho \left( \frac{\mu - 1}{\beta} \right) f \left( \frac{\mu}{\beta} \right) + (1 - \rho) \left( \frac{\mu - 1}{\beta} \right) f \left( \frac{(1 + \delta_m)\mu}{\beta} \right) \geq g \left( \frac{(1 + \delta_m)\mu}{\beta} \right).
\] (12)

Similarly, there exists \( \bar{\mu} \) which satisfies (12) with equality and the government budget constraint requires \( \mu \geq \bar{\mu} \) in equilibrium. This equilibrium case is shown as the area of 2 in Figure 1.

### 3.2.3 Partially pooling equilibrium with paper money (\( \mu = \eta \))

If \( \mu = \eta \), legal buyers are indifferent between using paper money or coins, but illegal buyers will only use paper money, so \( m^l, n^l, n^i > 0 \) and \( m^i = 0 \) in equilibrium. Legal buyers are indifferent between holding either form of money because they each provide the same rate of return. Therefore, the equilibrium allocation is determined by \( \eta \), which is similar to the pooling equilibrium with paper money. If both monies are actively used by legal trades in equilibrium there is an indeterminacy in the exchange rate between paper money and coin money because it can be arbitrarily chosen by the government.\(^{23}\) In the initial period, once a price pair \( (\phi_0, \psi_0) \) is determined with the initial supply of monies, \( M_0, N_0 \), the exchange rate is constant in the following periods because \( \mu = \eta \). Although both types of monies are used, the government budget constraint still requires \( \eta \geq \bar{\eta} \) regardless of the real money demand, \( m^l, n^l, \) and \( n^i \), because the sum of real money demand for legal buyers is determined by \( \eta \). This equilibrium case is shown as all points along line 3 in Figure 1.

### 3.2.4 Partially pooling equilibrium with coins (\( \alpha \mu = \eta \))

If \( \alpha \mu = \eta \), then illegal buyers are indifferent between paper money and coins, but legal buyers prefer to use coins, so \( m^i, n^i, m^l > 0 \) and \( n^l = 0 \) in equilibrium. Since illegal buyers are indifferent between holding either form of money, the equilibrium allocation in this case is determined by \( \mu \), which is similar to the pooling equilibrium with coins. However, note that the government budget constraint (9) can vary depending on whether illegal buyers use coins.

\(^{23}\)This indeterminacy in the relative price is the same result from Kareken and Wallace (1981).
or paper money. Since the rates of return for using two monies are equal for illegal buyers, \( x^i \) is maintained. However, the seigniorage revenue from illegal trades can increase when paper money is used, because \( \eta > \mu \) holds in this case. Suppose that a proportion of illegal traders, \( \theta \in [0, 1] \), use paper money whereas the rest of illegal trades use coins. By using the binding collateral constraints, \( \frac{\theta}{\mu}m^i = (1 - \theta)x^i \) and \( \frac{\eta}{\eta}n^i = \theta x^i \), we can re-write the government budget constraint (9) as

\[
\rho \left( \frac{\mu - 1}{\beta} \right) f \left( \frac{\mu}{\beta} \right) + (1 - \rho) \left[ \left( \frac{\mu - 1}{\beta} \right)(1 - \theta)f \left( \frac{(1 + \delta_m)\mu}{\beta} \right) + \left( \frac{\alpha\mu - 1}{\beta} \right) f \left( \frac{1 + \delta_m)\mu}{\beta} \right) \right] \geq g \left( \frac{(1 + \delta_m)\mu}{\beta} \right)
\]

Since \( \alpha > 1 \), the government budget constraint (13) can be relaxed as \( \theta \) increases. Therefore, this equilibrium case is shown as line 4 in Figure 1. Note that the line \( BC \) is feasible and corresponds to the cases with \( \theta \in [0, 1] \), but the allocation is indeterminate.

Since the rates of return for using two monies are equal for illegal buyers, there is also an indeterminacy in the exchange rate between two monies in the initial period. However, the exchange rate, i.e. the price of coin money divided by the price of paper money, increases over time by \( \alpha \).

### 3.2.5 Separating equilibrium \((\alpha \mu > \eta > \mu)\)

If \( \alpha \mu > \eta > \mu \), legal buyers prefer to use coins, but illegal buyers prefer to use paper money, so \( m^i, n^i > 0 \) and \( n^l = m^l = 0 \) in equilibrium, and \( x^l > x^i \) holds because \( \eta > \mu \). This is a separating equilibrium, but since both monies circulate we have an exchange rate between paper money and coin money. The initial exchange rate can be chosen arbitrarily, but the growth rate of the exchange rate can be between 1 and \( \alpha \). Using the first-order conditions, \( u'(x^l) = \frac{\mu}{\beta} \), \( u'(x^i) = \frac{\eta}{\beta} \), and the binding collateral constraints, \( \frac{\theta}{\mu}m^l = x^l \), \( \frac{\eta}{\eta}n^l = x^l \), we can re-write the government budget constraint (9) with policy variables, \( \mu \) and \( \eta \) as

\[
\rho \left( \frac{\mu - 1}{\beta} \right) f \left( \frac{\mu}{\beta} \right) + (1 - \rho) \left( \frac{\eta - 1}{\beta} \right) f \left( \frac{(1 + \delta_n)\eta}{\beta} \right) \geq g \left( \frac{(1 + \delta_n)\eta}{\beta} \right),
\]

(i4)
where $\mu$ and $\eta$ have a negative relationship shown as GBC(SE) curve in Figure 1. Note that the binding government budget constraint (14) intersects not only the point $A$ with $\mu = \eta = \bar{\eta}$, but also the point $C$ with $\alpha\mu = \alpha\bar{\mu} = \eta$ in Figure 1.

In the separating equilibrium each money is demanded by separated parties, legal and illegal buyers. Therefore, there is flexibility in the choice of the two monetary policy variables, $\mu$ and $\eta$, as long as the government budget constraint is satisfied. Moreover, in this separating equilibrium the feasible set of monetary policy variables with two monies is expanded to include the area of the triangle ABC. In the model, issuing two monies expands the feasible allocation set because doing so not only separates the types, but also makes a transfer from one type to the other possible by using different rates of return. This equilibrium case is shown as area 5 in Figure 1.

4 Welfare Analysis

4.1 Optimal Policy

In this subsection we compare the welfare implications of the feasible equilibria and try to determine the optimal monetary policy.

Lemma 1. The equilibrium allocations at $\mu > \eta \geq \bar{\eta}$ are the same as the one at $\mu = \eta \geq \bar{\eta}$, while the equilibrium allocation at $\mu \geq \bar{\mu}, \eta \geq \alpha\mu$ is also the same as the one at $\mu \geq \bar{\mu}, \eta = \alpha\mu$.

Lemma 2. Given the other variable, reducing $\mu$ or $\eta$ in a (partially) pooling equilibrium improves welfare.

Lemma 3. For given $\eta$, reducing $\mu$ in a separating equilibrium improves welfare.

Lemmas 1-3 show that welfare is maximized at a point or points on lines AC and BC in Figure 1. By Lemma 1, the monetary policy in areas 1 and 2 are equivalent with the monetary policy at the corresponding points on lines 3 and 4, respectively. Lemma 2 shows that the monetary policy on lines 3 and 4 can be dominated by the points A and B, respectively. By
Lemma 3, we know that the separating equilibrium allocations in the area (5) are suboptimal to the equilibrium allocations on lines AC, BC and 4, which is dominated by the point B. Therefore, we can compare the welfare of the equilibrium allocations on lines AC and BC to find out the optimal monetary policy.

**Lemma 4.** *When the proportion of illegal trades using paper money, \( \theta \), increases in the partially pooling equilibrium with coins, welfare improves.*

Lemma 4 shows that point C dominates the allocations on line BC in the partially pooling equilibrium with coins. However, there is an indeterminacy in the allocations corresponding to the monetary policy on line BC. These allocations could not be supported as an equilibrium, because it depends on the demands of illegal traders for paper money and coins, which are indifferent for illegal trades in the partially pooling equilibrium. Thus, when the monetary policy with \( \eta = (1 + \delta_m)\mu \) is implemented on line BC, the government budget constraint could be violated by the coin usage of illegal traders. Therefore, even though point C dominates point B in terms of welfare, monetary policy is capable of implementing point B, but not point C, with certainty.

Note that point A represents monetary policy with both monies while point B can be interpreted as the monetary policy without the paper money. Eliminating paper money is suboptimal here because if we implement monetary policy with \( \eta = (1 + \delta_m)\mu \) and illegal traders use paper money instead of coins in point C, then we can support more legal and illegal trade by collecting more seigniorage revenue. This result seems to be contrary to the intuition of Rogoff (2016), who argues that eliminating large bills can improve welfare if the inefficiency of illegal trade is sufficiently large. However, we can understand his intuition with the social welfare function of \( W = \rho \{ u(x^l) - x^l \} \), in which illegal trades are not beneficial at all from a social perspective. In this case legal buyers consume more at point B than point A, because of \( \bar{\mu} < \bar{\eta} \) as long as \( \delta_m > \delta_n \) holds.

**Proposition 1.** *If \( \delta_n \) is sufficiently large, i.e. \( \delta_n \geq \hat{\delta}_n \), then welfare is maximized at the pooling equilibrium with paper money. If \( \delta_n < \hat{\delta}_n \), the pooling equilibrium with paper money is suboptimal.*
Proposition 2. When $\delta_n < \hat{\delta}_n$, if $\delta_m > \hat{\delta}_m$, then welfare is maximized at a separating equilibrium. Otherwise, if $\delta_m \leq \hat{\delta}_m$, the optimal policy is to implement a partially pooling equilibrium with coins.

Proposition 1 implies that given the marginal social cost of illegal trades, $c(x^i)$, if $\delta_n$ is sufficiently small, then illegal traders generate the negative impact without paying the proper cost, so the pooling equilibrium with paper money is inefficient. In this case, raising the rate of return on coins, $\frac{1}{\mu}$, and reducing the rate of return on paper money, $\frac{1}{\eta}$, can improve welfare. It can directly reduce the illegal trading amount, $x^i$, and the social cost, $c(x^i)$. Moreover, with the saving in government spending, it can make a distributional transfer to legal buyers which will lead $x^l$ to increase.

Proposition 2 shows that given the marginal social cost of illegal trades, $c(x^i)$, if $\delta_n$ is sufficiently small and $\delta_m$ is sufficiently large, then the social welfare indifference curve has a tangency on the line AC as shown in Figure 1 and the optimal monetary policy can be found between points A and C. Consider the allocation at the point C, where legal buyers use coins and illegal buyers use paper money in a separating equilibrium. Since $\delta_m$ is sufficiently large, the illegal trades and the social cost are less than their optimal levels. Thus, raising the rate of return on paper money and reducing the rate of return on coins at the point C can improve welfare by increasing $x^i$ and decreasing $x^l$.

In other words, Proposition 2 implies that if $\delta_m$ is also sufficiently small, then point C is optimal. Notice that point C can be achieved when illegal trades use paper money in a partially pooling equilibrium. Therefore, although the hiding cost for illegal traders is not sufficient to reduce their activities, it would be beneficial to keep paper money in circulation with a lower rate of return, because we can collect the inflation tax from the illegal trades more efficiently with paper money and make a transfer to the legal traders.

If our welfare function follows $W = \rho\{u(x^l) - x^i\}$ as discussed above, point C is the optimal monetary policy regardless of the level of social cost and illegal trade, because it is always beneficial if we can make a transfer from the illegal traders to legal traders. In this respect, the expansion of the feasible allocation set is an advantage of using two different types of
payment methods. The triangle area ABC in Figure 1 is infeasible under pooling equilibrium, but feasible under separating equilibrium. This result is because separating types of buyers with two different payment methods allow the government to collect seigniorage from one type and make a transfer to the other type even with anonymity.

Finally, note that in this model $\delta_m$ can be related to the denomination structure. For example, given the highest denomination bills, e.g. 500 Euros, if the second highest bills are changed from 100 Euros to 10 Euros, then it becomes more costly to hide illegal trades and so $\delta_m$ increases.

4.2 Discussion

We have two additional points to discuss about the results of the model. First, in a separating equilibrium the choices of payment methods reveal types and allow the government to make a transfer from those engaging in illegal trade to those engaged in legal trade. Note that this type of revelation does not rely on monitoring or recording devices. In the model, we do not know who uses which type of money, but we can treat them differently by their types. Legal trade is more socially desirable while illegal activities are less so. Thus, given heterogeneous preferences over payment methods, collecting more taxes from illegal traders and providing a transfer to legal traders is beneficial. So, the government can improve welfare by adjusting the exchange rate between the two monies or pay an interest on one type of money under anonymity.

Second, a practical implementation of this plan would be to allow for two types of monies to circulate with strictly positive interest paid on one type of money. Alternatively, this policy could be implemented by allowing the exchange rate between these two types of monies to change over time. In each case, the monetary authority can use monetary policy to facilitate the transfer described in our model. For example, suppose that we apply this policy to two types of money:

---

24If one type of money can be monitored, we may need to consider unsecured credit along with two monies in the model, which is beyond the scope of this paper.

25In this type of model money is neutral, but not super-neutral because money is divisible and the rate of return on money determines the type of money and the quantity of trade. Similarly, the exchange rate between coins and paper money in separating equilibrium is also neutral, but not super-neutral. So the initial level of exchange rate can be indeterminate as shown in Kareken and Wallace (1981), but the growth rate of exchange rate can be a policy variable to affect equilibrium allocations.
small denomination bills and large denomination bills. If nominal interest payments on small bills are feasible, as shown in Andolfatto (2010), then a strictly positive nominal interest rate on small bills can support the same equilibrium allocation instead of changing the exchange rate between large bills and small bills.

5 An Endogenous Government Budget Constraint

In this section, we study whether the previous results hold when the government authority is subject to the same degree of limited commitment. The government cannot enforce the lump-sum taxes, but can collect the lump-sum taxes voluntarily from the buyers who want to keep using the monies. We assume that the government suggests an amount of lump-sum tax for buyers to pay voluntarily. If the tax payment is not received, then the government will not issue either money any longer, so the agents could be in autarky. We describe the feasible allocations for pooling equilibria with paper money, coins, and the separating equilibrium. Then, we consider whether a separating equilibrium can still maximize welfare even if agents are not able to be forced to pay the lump-sum tax.

5.1 Pooling equilibrium with paper money

For type $j \in \{l, i\}$ agents,

$$\tau - (1 + \delta n \mathbb{I}_{j=i}) n^j + u(x^j) \geq 0,$$

is required to pay the (negative) lump-sum transfer voluntarily in the period $t CM$. Otherwise, the type $j$ agents would opt out in favor of autarky. Thus, the tax incentive constraint (15) implies that a trading gain of an agent $j$ must be greater than the lump-sum tax.

Using the first-order conditions, $\frac{(1+\delta n \mathbb{I}_{j=i})n}{\beta} = u'(x^j)$, and the binding constraint, $\frac{\beta}{\eta} n^j = x^j$, note that $(1 + \delta n \mathbb{I}_{n_i>0}) n^j$ can be transformed into $x^i u'(x^j)$, which can be rewritten as $(1 - \gamma) u(x^j)$ with the CRRA utility function. Since $x^i < x^l$ in equilibrium, the incentive constraint for type $i$ agents always bind instead of the incentive constraint for type $l$. Thus,
we focus only on type \( l \) agents. The transfer, \( \tau \), includes the seigniorage revenue minus the government spending,
\[
\tau = \left(1 - \frac{1}{\eta}\right)\{\rho n^l + (1 - \rho) n^i\} - c(x^i). \quad (16)
\]

After plugging (16) into (15), by using the first-order conditions and the binding constraints we can transform (15) into
\[
\rho x^l \left(u'(x^l) - \frac{1}{\beta}\right) + (1 - \rho) x^i \left(u'(x^i) \frac{1 + \delta_n}{1 - \beta}\right) - c(x^i) + u(x^i) - x^i u'(x^i) \geq 0, \quad (17)
\]
where the tax incentive constraint (17) is represented as a sum of the seigniorage from both types, the social cost of illegal trades, and the trading gain of type \( i \) agents. Then, with the CRRA utility function, (17) can be rearranged into
\[
\left(\frac{1 - \gamma}{1 + \delta_n} + \frac{\gamma}{\rho_n}\right) u(x^i) \geq \frac{1}{\rho_n} c(x^i) + \frac{1}{\beta} x^i, \quad (18)
\]
where \( \rho_n := \rho (1 + \delta_n) \frac{1}{\beta} + 1 - \rho \geq 1. \)

Define \( \bar{x} \) as \( x^i \) which satisfies (18) with equality. Since the left-hand side is strictly concave and the right-hand side is strictly convex, if \( \beta < \bar{\beta} \) then \( \bar{x} < x^* \), when \( \bar{\beta} \) satisfies with \( \left(\frac{1 - \gamma}{1 + \delta_n} + \frac{\gamma}{\rho_n}\right) u(x^*) = \frac{1}{\rho_n} c(x^*) + \frac{1}{\beta} x^* \). In this case the monetary policy is limited to \( \eta \geq \bar{\eta} \) where \( u'(\bar{x}) = \bar{\eta} \).

5.2 Pooling equilibrium with coins

In the pooling equilibrium with coins, the tax incentive constraints for type \( j \) is
\[
\tau - (1 + \delta_m)m^j \mathbb{I}_{(j = \bar{i})} + u(x^j) \geq 0. \quad (19)
\]
By plugging \( \tau = \left(1 - \frac{1}{\mu}\right)\{\rho m^l + (1 - \rho)m^i\} - c(x^i) \) and using the first-order conditions, \( \frac{(1+\delta_n(1-\delta_m))\mu}{\beta} = u'(x^j) \), and the binding constraint, \( \frac{\beta}{\mu} m^j = x^j \), we can rewrite (19) as

\[
\rho x^l \left( u'(x^l) - \frac{1}{\beta} \right) + (1 - \rho) x^i \left( \frac{u'(x^i)}{1 + \delta_m} - \frac{1}{\beta} \right) - x^i u'(x^i) + u(x^i) - c(x^i) \geq 0. \tag{20}
\]

Notice that comparing to the pooling equilibrium with paper money, the seigniorage revenue from the illegal traders becomes smaller because the high hiding cost with \( \delta_m > \delta_n \) reduces the demand coins further. Moreover, since \( x^i < x^l \) holds in equilibrium, the incentive constraint for type \( i \) agents always binds here. With the CRRA utility function, the incentive constraint (20) can be reduced into

\[
\left( \frac{1 - \gamma}{1 + \delta_m} + \gamma \rho_m \right) u(x^i) \geq \frac{1}{\rho_m} c(x^i) + \frac{1}{\beta} x^i. \tag{21}
\]

where \( \rho_m := \rho(1 + \delta_m)^{\frac{1}{\gamma}} + 1 - \rho > \rho_n \geq 1. \)

We define \( \hat{x} \) as \( x^i \) which satisfies (21) with equality. If \( \beta \) is sufficiently small, i.e. \( \beta \leq \tilde{\beta} \), then \( \hat{x} < x^* \) also holds because \( \hat{x} < x^* \) and \( \rho_m > \rho_n \). So the monetary policy can be limited at the lower bound \( \hat{\mu} \), which can be defined as \( u'(\hat{x}) = \frac{\rho}{\beta} \).

### 5.3 Separating equilibrium

In separating equilibrium we need to consider four tax incentive constraints,

\[
\begin{align*}
\tau - m^l + u(x^i) & \geq 0, \\
\tau - m^l + u(x^i) & \geq \tau - n^l + u(x^l), \tag{22} \\
\tau - (1 + \delta_n)n^i + u(x^i) & \geq 0, \\
\tau - (1 + \delta_n)n^i + u(x^i) & \geq \tau - (1 + \delta_m)m^i + u(x^i),
\end{align*}
\]

because the government can eliminate not only both monies, but also only one type of money. Note that the second and fourth constraints do not bind by definition of separating equilibrium.

Similar to the previous cases, by plugging \( \tau = \left(1 - \frac{1}{\mu}\right)\rho m^l + (1 - \frac{1}{\nu})(1 - \rho)n^i - c(x^i) \) and the
equilibrium conditions into the first and third ones, we have

\[
\rho x^l \left( u'(x^l) - \frac{1}{\beta} \right) + (1 - \rho) x^i \left( \frac{u'(x^i)}{1 + \delta_n} - \frac{1}{\beta} \right) - \frac{c(x^i)}{1 + \delta_n} + u(x^j) - x^j u'(x^j) \geq 0. \tag{23}
\]

Given the first-order conditions, \( u'(x^l) = \frac{\mu}{\beta} \) and \( u'(x^i) = \frac{(1 + \delta_n) \eta}{\beta} \), in separating equilibrium \( \frac{1 + \delta_m}{1 + \delta_m} \mu > \eta > \mu \) holds. Therefore, \( x^i < x^l \) always holds in equilibrium and only the tax incentive constraint for type \( i \) binds.

We assume that the trading gain of illegal traders minus the social cost decreases in \( x^i \), to obtain a negative relationship between \( x^l \) and \( x^i \) when the tax incentive constraint (23) binds. This assumption is reasonable, because otherwise illegal trades must be promoted for society all the time.

Both pooling equilibrium allocations are also feasible with (23), because (23) is the same as (17) and more relaxed than (20) with \( \delta_m > \delta_n \). Thus, in Figure 2, the pooling equilibrium with paper money, the point A, and the pooling equilibrium with coins, the point B, are available by the monetary policy along with the tax incentive constraint, \( IC' \).

[Figure 2 here]

Lemmas 1-4 also apply here with the same logic and we can have the similar result as shown in Propositions 1-2.

**Corollary 1.** When \( \delta_n \geq \tilde{\delta}_n \), welfare is maximized at a pooling equilibrium with paper money. When \( \delta_n < \tilde{\delta}_n \), if \( \delta_m > \tilde{\delta}_m \), then welfare is maximized at a separating equilibrium. Otherwise, if \( \delta_m \leq \tilde{\delta}_m \), then the optimal monetary policy is to implement a partially pooling equilibrium with coins.

Corollary 1 confirms that if the tax incentive constraint binds with the sufficiently small time preference, \( \beta \), then welfare can still be maximized at a separating equilibrium under limited commitment.
5.4 Discussion

In pooling equilibrium when $\beta$ is sufficiently small, the tax incentive constraint could bind: Agents prefer current consumption to future consumption, so they avoid paying a lump-sum tax even though it improves the welfare of the whole society. Given the binding constraint, we can increase the consumption of the legal traders by raising the rate of return on coins. It will reduce the seigniorage generated by the legal traders, but we can offset this seigniorage loss by reducing the rate of return on paper money, because the increase in the seigniorage generated by the illegal traders plus the reduced social cost dominates the decrease in the trading gain of an illegal trader. Thus, by transferring seigniorage implicitly, we can redistribute the trade surplus between the two types of agents as shown in the section 4.

6 Conclusion

We show that eliminating a money preferred by illegal traders can be suboptimal if the transaction cost of the alternative money for illegal traders is sufficiently large. In particular, our model shows that if illegal trade reduces social welfare, then it is optimal to tax illegal traders implicitly to finance a transfer to legal traders. In our model, this is achieved by using monetary policy to create an equilibrium in which the choice of currency is dependent on one’s type. By doing so, policy can generate more seigniorage from illegal traders and also finance a transfer to legal traders, even though the illegal trades are not observable.

Nonetheless, our paper should not be the final word on the topic. One opportunity for future research would be to examine the issue of tax avoidance, such as whether tax revenue would increase sufficiently to replace lost seigniorage if cash was eliminated. Additionally, the results of our paper show that exchange rates or transaction costs between different payment methods can be an alternative policy tool in the presence of heterogeneous preferences over alternative transaction methods. This might be related to a question raised by Stiglitz (2017) that asks whether we can control aggregate demand by adjusting the cost of different payment
methods rather than changing the overall price level. Likewise, Agarwal and Kimball (2015) suggest that negative interest rates could be feasible by adjusting exchange rate between two payment methods, e-money and currency. Work on this exchange rate channel could also be worthwhile in the future.

\[26\] Instead of implementing -2% in reserves, i.e. e-money in the paper, reducing the exchange rate of currency by 2% continuously, like 1, 0.98, 0.96, ..., can satisfy no-arbitrage condition between currency and e-money. This mechanism is similar to our separating equilibrium case in which exchange rates grows at a fixed rate, but in our separating equilibrium the no-arbitrage condition does not hold because we have a heterogeneity in types.
References


Appendix

Proof of Lemma 1:

Proof. In a pooling equilibrium with paper money at $\mu > \eta \geq \bar{\eta}$, coin money is not used, thus $\mu$ is irrelevant. In the equilibrium with $\mu = \eta \geq \bar{\eta}$ coin money can be used along with paper money, but the allocation is maintained because coin money can be replaced by paper money given the same rate of return. Similar logic applies for the case of the pooling equilibrium with coins at $\mu \geq \bar{\mu}, \eta \geq \alpha \mu$.

Proof of Lemma 2:

Proof. Since $u'(x^l) = \frac{\eta}{\beta}$ and $u'(x^i) = \frac{(1+\delta_n)\eta}{\beta}$ holds in a (partially) pooling equilibrium with paper money, the welfare effect of reducing $\eta$ is

$$\frac{\partial W}{\partial \eta} = \rho \{ u'(x^l) - 1 \} \frac{\partial x^l}{\partial \eta} + (1 - \rho) \{ u'(x^i) - 1 \} \frac{\partial x^i}{\partial \eta},$$

where $\frac{\partial x^l}{\partial \eta} = \frac{1}{\beta u'(x^l)} < 0$ and $\frac{\partial x^i}{\partial \eta} = \frac{1+\delta_n}{\beta u'(x^i)} < 0$. Thus, $\frac{\partial W}{\partial \eta} < 0$. Similarly, $\frac{\partial W}{\partial \mu} < 0$ because

$$\frac{\partial W}{\partial \mu} = \rho \{ u'(x^l) - 1 \} \frac{\partial x^l}{\partial \mu}$$

where $\frac{\partial x^l}{\partial \mu} = \frac{1}{\beta u''(x^l)} < 0$.

In case of a (partially) pooling equilibrium with coins, we have $u'(x^l) = \frac{\mu}{\beta}; u'(x^i) = \frac{(1+\delta_m)\mu}{\beta} = \frac{(1+\delta_n)\eta}{\beta}$. The welfare effect of reducing $\mu$ is also strictly negative as

$$\frac{\partial W}{\partial \mu} = \rho \{ u'(x^l) - 1 \} \frac{\partial x^l}{\partial \mu} \frac{\partial W}{\partial \eta} = \frac{\rho \{ u'(x^l) - 1 \} \frac{\partial x^l}{\partial \mu}}{\beta u''(x^l)} + (1 - \rho) \frac{(1+\delta_m)(u'(x^i) - 1)}{\beta u''(x^i)} < 0.$$

Similarly, $\frac{\partial W}{\partial \eta} < 0$ because $\frac{\partial W}{\partial \eta} = \rho \{ u'(x^i) - 1 \} \frac{\partial x^i}{\partial \eta}$ where $\frac{\partial x^i}{\partial \eta} = \frac{1+\delta_n}{\beta u''(x^i)} < 0$.

Proof of Lemma 3:

Proof. Since $x^l$ is independently determined by $u'(x^l) = \frac{\mu}{\beta}$ in a separating equilibrium, the welfare effect is strictly negative by reducing $\mu$ as

$$\frac{\partial W}{\partial \mu} = \rho \{ u'(x^l) - 1 \} \frac{\partial x^l}{\partial \mu} = \rho \frac{\{ u'(x^l) - 1 \}}{\beta u''(x^l)} < 0.$$

Proof of Lemma 4:

Proof. In the partially pooling equilibrium with coins, if $\theta$ increases then the government budget constraint (13) can be relaxed because $\alpha > 1$. Therefore, $\mu$ can go down and both $(x^l, x^i)$ will increase by the first-order conditions, $u'(x^l) = \frac{\mu}{\beta}$ and $u'(x^i) = \frac{\alpha \mu}{\beta}$.

Proof of Proposition 1:
Proof. Suppose that there is a pooling equilibrium allocation with paper money, \((x^p_s, x^p_i)\), at \(\mu = \eta = \bar{\eta}\), shown as point A in Figure 1. Our claim is that reducing \(\mu\) and raising \(\eta\), in which the allocation moves towards the point C along the line AC, can improve welfare, if \(\delta_n\) is sufficiently small. Given the first-order conditions, \(u'(x^i) = \frac{u'}{\beta}\) and \(u'(x^i) = \frac{(1+\delta_n)\eta}{\beta}\), and the government budget constraint (14), \(x^i\) increases and \(x^i\) decreases by reducing \(\mu\) and raising \(\eta\) on the line AC. Using the first-order conditions, we can re-write the binding government budget constraint (14) as

\[
\rho \left( u'(x^i) - \frac{1}{\beta} \right) x^i + (1 - \rho) \left( \frac{u'(x^i)}{1+\delta_n} - \frac{1}{\beta} \right) x^i - c(x^i) = 0. \tag{24}
\]

Since the equilibrium allocation \((x^l, x^i)\) is determined by the first-order conditions and (24), the welfare effect of increasing \(x^l\) and decreasing \(x^i\) can be described as

\[
\frac{\partial W}{\partial x^l} = \rho\{u'(x^l) - 1\} + (1 - \rho)\{u'(x^i) - 1\} \frac{\partial x^i}{\partial x^l} \mid G. \tag{25}
\]

Thus, if \(\frac{\partial x^i}{\partial x^l} \mid G > -\frac{\rho (u'(x^i) - 1)}{(1 - \rho) (u'(x^i) - 1)}\) then \(\frac{\partial W}{\partial x^l} > 0\) in (25). We can derive \(\frac{\partial x^i}{\partial x^l} \mid G\) from (24) by using \(-\frac{x^i u''(x)}{u'(x)} = \gamma\) and show the inequality in (26) holds as below when \(\delta_n = 0\): When \(\delta_n = 0\), notice that \(x^p_s = x^p_i, c'(x^p_i) > 0\), and \((1 - \gamma)u'(x^i) - \frac{1}{\beta} < 0\) must hold to confirm that the seigniorage revenue increases in \(\eta\), because \((\frac{2}{\gamma} - 1) f(\frac{\gamma}{\beta}) = (u'(x^i) - \frac{1}{\beta}) x^i = (1 - \gamma) u(x^i) - \frac{1}{\beta} x^i\).

\[
\frac{\partial x^i}{\partial x^l} \mid G = -\frac{\rho (1 - \gamma) u'(x^p_i) - \frac{1}{\beta}}{(1 - \rho) (1 - \gamma) u'(x^p_i) - \frac{1}{\beta} - c'(x^p_i)} 
= -\frac{\rho (u'(x^p_i) - 1)}{(1 - \rho) (u'(x^p_i) - 1)} \tag{26}
\]

Note that when \(\delta_n\) increases, \(x^p_i\) decreases by the first-order condition and \(c'(x^p_i)\) approaches to zero. Thus, there exists a threshold \(\hat{\delta}_n\), above which \(\frac{\partial x^i}{\partial x^l} \mid G \leq -\frac{\rho (u'(x^p_i) - 1)}{(1 - \rho) (u'(x^p_i) - 1)}\) holds in (26), and \(\frac{\partial W}{\partial x^l} \leq 0\) in (25). Hence, if \(\delta_n \geq \hat{\delta}_n\) then welfare is maximized at a pooling equilibrium with paper money. \(\square\)

Proof of Proposition 2:

Proof. Suppose that there is a separating equilibrium allocation, \((x^d_s, x^d_i)\), at \(\mu = \bar{\mu}\) and \(\eta = \alpha \bar{\mu}\), described as the point C in Figure 1. Our claim is that raising \(\mu\) and reducing \(\eta\), where the allocation moves towards the point A, can improve welfare, if \(\delta_m\) is sufficiently large. Similarly to the proof of the Proposition 1, we can compare \(\frac{\partial x^i}{\partial x^l} \mid G\) with \(\frac{\partial x^i}{\partial x^l} \mid W = -\frac{\rho (u'(x^d_i) - 1)}{(1 - \rho) (u'(x^d_i) - 1)}\) at the allocation, \((x^d_s, x^d_i)\), with \(\delta_n = 0\) as below.

\[
\frac{\partial x^i}{\partial x^l} \mid G = -\frac{\rho ((1 - \gamma) u'(x^d_i) - \frac{1}{\beta})}{(1 - \rho) ((1 - \gamma) u'(x^d_i) - \frac{1}{\beta}) - c'(x^d_i)} < -\frac{\rho (u'(x^d_i) - 1)}{(1 - \rho) (u'(x^d_i) - 1)} \tag{27}
\]

Given the first-order condition, \(u'(x^i) = (1 + \delta_m)u'(x^i)\), \(u'(x^i)\) can be replaced with \(\frac{u'(x^i)}{1 + \delta_m}\) in (27), so the inequality of (27) depends on \(\delta_m\) and \(x^i\). Note that when \(\delta_m\) increases, \(x^i\) decreases in equilibrium. Therefore, \(c'(x^i)\) can approach zero, when \(x^i\) decreases to zero. If \(c'(x^i)\)
approaches to zero, then the right-hand side of (27) is greater than the left-hand side of (27) with $\delta_m > 0$ and $x^l > x^i$. If $\delta_m = 0$, then the right-hand side of (27) is smaller than the left-hand side of (27) because $c'(x^i) > 0$ and $x^l = x^i$ in equilibrium. Thus, there exists a threshold, $\hat{\delta}_m > 0$, above which welfare is maximized at a separating equilibrium. \hfill \square

**Proof of Corollary 1:**

*Proof.* Similarly to the proof of the Propositions 1-2, given the CRRA utility function we can compare $\frac{\partial x^i}{\partial x^l} |_{IC}$ with $\frac{\partial x^i}{\partial x^l} |_W$ at the points A and B in Figure 2, respectively, as below.

\[
\frac{\partial x^i}{\partial x^l} |_{IC} = -\frac{\rho\{(1-\gamma)u'(x^l) - \frac{1}{\beta}\}}{(1-\rho)\{(1-\gamma)\frac{u'(x^i)}{1+\delta_n} - \frac{1}{\beta}\}} c'(x^i) + \gamma u'(x^i) \geq \frac{\partial x^i}{\partial x^l} |_W
\]  

(28)

Suppose that there is a pooling equilibrium with paper money at the point A with $\mu = \eta = \tilde{\eta}$. If $\delta_n$ approaches to zero, then $\frac{\partial x^i}{\partial x^l} |_{IC} > \frac{\partial x^i}{\partial x^l} |_W$ in (28) because $x^l = x^i$ and $\gamma u'(x^i) < c'(x^i)$. Therefore, there exists a threshold of $\hat{\delta}_n$, above which welfare is maximized at a pooling equilibrium with paper money.

Now suppose that there is a separating equilibrium allocation at the point C with $\mu = \hat{\mu}$ and $\eta = \alpha \hat{\mu}$, described as the point C in Figure 1. Since $\gamma u'(x^i) < c'(x^i)$, we can follow the proof of the Proposition 2 and there exists a threshold of $\hat{\delta}_m$, above which welfare is maximized at a separating equilibrium.

Finally, when the tax incentive constraint binds, the pooling equilibrium with coins is strictly dominated by the separating equilibrium with $\mu = \hat{\mu}$ and $\eta = \alpha \hat{\mu}$, because $\delta_m$ in (20) is greater than $\delta_n$ in (23). \hfill \square
Figure 1: Welfare Analysis

Figure 2: Endogenous Incentive Constraint: Type $i$ binds